PSEUDO-INDEX OF FANO MANIFOLDS AND SMOOTH BLOW-UPS

LAURENT BONAVERO

Abstract. Suppose $\pi : X \to Y$ is a smooth blow-up along a submanifold Z of Y between complex Fano manifolds X and Y of pseudo-indices i_X and i_Y respectively (recall that i_X is defined by $i_X := \min\{-K_X \cdot C \,|\, C \text{ is a rational curve of } X\}$). We prove that $i_X \leq i_Y$ if $2\dim(Z) < \dim(Y) + i_Y - 1$ and show that this result is optimal by classifying the "boundary" cases. As expected, these results are obtained by studying rational curves on X and Y.

1. Statement of the results

1.1. **Introduction.** When studying surjective morphisms $f : X \to Y$ between smooth Fano manifolds X and Y of the same dimension, one generally observes that the anti-canonical bundle $-K_Y$ of Y is "more positive" than the anti-canonical bundle $-K_X$ of X, one of the most important results in this direction being the famous theorem of Lazarsfeld [La83] stating that if $\mathbb{P}^n \to Y$ is a surjective morphism from $\mathbb{P}^n$ to an n-dimensional manifold Y, then $Y \simeq \mathbb{P}^n$.

For a Fano manifold X (i.e., a complex manifold with ample anti-canonical line bundle $-K_X$), one defines two integers called the index r_X and the pseudo-index i_X of X by

$$r_X := \max\{m \in \mathbb{N} \,|\, -K_X = mL \text{ with } L \in \mathrm{Pic}(X)\}$$

and

$$i_X := \min\{-K_X \cdot C \,|\, C \text{ is a rational curve of } X\}.$$

Of course, i_X is a multiple of r_X and many results are known for these numbers. Among others, Fano manifolds of dimension n with large index (namely bigger than $n-2$) are classified (see [IP99] for a complete survey on Fano manifolds), the situation being much more complicated for the pseudo-index: one knows that $i_X \leq n+1$ by Mori theory, equality holding if and only if $X \simeq \mathbb{P}^n$ [CMS00].

1.2. **The main result.** Let us start with an easy remark: let Y be a complex manifold of dimension n, let Z be a connected submanifold of Y, let $X := B_Z(Y)$ be the blow-up of Y with center Z and let E be the exceptional divisor of $\pi : X \to Y$. We classically have

$$H^2(X, \mathbb{Z}) \simeq H^2(Y, \mathbb{Z}) \oplus \mathbb{Z} \cdot E \text{ and } K_X = \pi^* K_Y + (n - \dim(Z) - 1)E.$$

Therefore, if both Y and X are Fano, r_X is equal to the greatest common divisor of r_Y and $n - \dim(Z) - 1$, which implies in particular that $r_X \leq r_Y$ and confirms the philosophy described above.

In this Note, we study the behaviour of the pseudo-index with respect to smooth blow-ups. Quite surprisingly, this behaviour depends on the dimension of the center of the blow-up. Our precise results are the following.

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Theorem 1. *Let Y be a complex manifold of dimension n, let Z be a connected submanifold of Y and let $X := B_Z(Y)$ be the blow-up of Y with center Z. Suppose both Y and X are Fano.*

(i) *If $2\dim(Z) < n + i_Y - 1$, then $i_X \leq i_Y$,*
(ii) *if $2\dim(Z) = n + i_Y - 1$ and $i_Y \geq 2$, then $i_X \leq i_Y$,*
(iii) *if $\dim(Z) < n/2$, then $i_X \leq i_Y$.*

Of course, (iii) is an obvious consequence of (i) since $i_Y \geq 1$. This result says that the pseudo-index has the "expected behaviour" when the center of the blow-up has small dimension. Remark that the case where $\dim(Z) = 0$ could be proved by looking at the classification given in [BCW02] and the case where $\dim(Z) = 1$ is Proposition 3.7 of [BCDD03].
Let us now give an example where i_X is bigger than i_Y. In the following proposition (as in the whole paper), we do not follow Grothendieck's convention: $\mathbb{P}(V)$ denotes the projective space of *lines* of the vector space V.

Proposition 1. *Let $n := 2m$ be an even integer, let $\mathcal{E}$ be the following rank $m+1$ vector bundle over $\mathbb{P}^m$:*

$$\mathcal{E} = \mathcal{O}_{\mathbb{P}^m}^{\oplus m} \oplus \mathcal{O}_{\mathbb{P}^m}(1)$$

and let Y_n be the n-dimensional manifold $Y_n = \mathbb{P}(\mathcal{E})$. The trivial rank m-subbundle of $\mathcal{E}$ defines a submanifold Z_m isomorphic to $\mathbb{P}^{n/2}$ with normal bundle N_{Z_m/Y_n} isomorphic to $\mathcal{O}_{\mathbb{P}^m}(-1)^{\oplus m}$. Finally, let $\pi_n : X_n = B_{Z_m}(Y_n) \to Y_n$ be the blow-up of Y_n along Z_m. Then Y_n and X_n are Fano manifolds of dimension n if $n \geq 4$. Moreover $i_{Y_n} = 1$, $i_{X_4} = 1$ and $i_{X_n} = 2$ if $n \geq 6$.

Therefore, the inequalities of Theorem 1 are optimal: for any $n = 2m \geq 6$, $\pi_n : X_n = B_{Z_m}(Y_n) \to Y_n$ is a blow-up with smooth connected center between Fano manifolds with $\dim(Z_m) = \dim(X_n)/2$ and $i_{X_n} > i_{Y_n}$.

Proof of Proposition 1. Since X_n and Y_n are naturally toric manifolds, it is enough to compute the anti-canonical degree of invariant (rational) curves. If d is a line contained in Z_m, then $-K_{Y_n} \cdot d = 1$, which gives $i_{Y_n} = 1$. The Fano manifold X_n is isomorphic to the $\mathbb{P}^1$-bundle $\mathbb{P}_{\mathbb{P}^{m-1}\times\mathbb{P}^m}(\mathcal{O}_{\mathbb{P}^{m-1}\times\mathbb{P}^m} \oplus \mathcal{O}_{\mathbb{P}^{m-1}\times\mathbb{P}^m}(1,1))$ over $\mathbb{P}^{m-1} \times \mathbb{P}^m$ hence $i_{X_n} \leq 2$ (the $\mathbb{P}^1$-fibers having anti-canonical degree equal to 2). Let $E \simeq \mathbb{P}^{m-1} \times \mathbb{P}^m$ be the exceptional divisor of π_n: the lines contained in a $\mathbb{P}^{m-1} \times \{*\} \subset E$ have anti-canonical degree equal to $m-1$, hence $i_{X_4} = 1$ and $i_{X_n} = 2$ if $n \geq 6$. ■

Remark: for $n \geq 8$, the previous computations show that the rational curves in X_n of minimal anti-canonical degree are not mapped by π_n to curves of minimal anti-canonical degree in Y_n.

Let us now discuss in more details the optimality of Theorem 1 by classifying the "boundary cases".

Theorem 2. *Let $\pi : X \to Y$ be a blow-up with smooth connected center Z between Fano manifolds X and Y of dimension n. If $2\dim(Z) = n + i_Y - 1$ then $i_X \leq i_Y$ unless $n \geq 6$ is even, $X = X_n$, $Y = Y_n$ and $\pi = \pi_n$.*

1.3. **Some consequences.** The results above have the following consequences when the pseudo-index of Y is large or in low dimensions.

Corollary 1. *Let $\pi : X \to Y$ be a blow-up with smooth connected center between Fano manifolds X and Y of dimension n.*

(i) *If* $i_Y > n/3 - 1$, *then* $i_X \leq i_Y$.
(ii) *If* $i_Y = n/3 - 1$, *then* $i_X \leq i_Y$ *unless* $n = 6$, $X = X_6$, $Y = Y_6$ *and* $\pi = \pi_6$.

Proof of Corollary 1.
Proof of (i). Suppose by contradiction that $i_X > i_Y$. Then by Theorem 1(i), $2\dim(Z) \geq n + i_Y - 1$. But the lines contained in the non-trivial fibers of the blow-up are rational curves of anti-canonical degree $n - 1 - \dim(Z)$, therefore $n - 1 - \dim(Z) \geq i_X > i_Y$, hence

$$n - 1 - i_Y \geq \dim(Z) + 1 \geq n/2 + i_Y/2 + 1/2$$

and $i_Y \leq n/3 - 1$, a contradiction.
Proof of (ii). Suppose that $i_Y = n/3 - 1$ and that $i_X > i_Y$. The previous computations implies that every inequality occuring in the proof of (i) is an equality. In particular, one has $2\dim(Z) = n + i_Y - 1$. Therefore, Theorem 2 implies that n is even and $Y = Y_n$. In particular, $i_Y = 1 = n/3 - 1$, hence $n = 6$, which ends the proof. ■

Remark that according to the generalised Mukai conjecture, as stated and studied in [BCDD03], Fano manifolds Y of dimension $n \geq 6$ with $i_Y > n/3 - 1$ should have Picard number ρ_Y satisfying $\rho_Y < \dfrac{3n}{n-6}$. Corollary 1 has the immediate following corollary.

Corollary 2. *Let* $\pi : X \to Y$ *be a blow-up with smooth connected center between Fano manifolds* X *and* Y *of dimension* n.
(i) *If* $n \leq 5$, *then* $i_X \leq i_Y$.
(ii) *If* $n = 6$, *then* $i_X \leq i_Y$ *unless* $X = X_6$, $Y = Y_6$ *and* $\pi = \pi_6$.

2. Proofs

2.1. **Proof of Theorem 1.** It is enough to prove assertion (i), since (iii) is an obvious consequence of (i) and (ii) is an immediate consequence of Theorem 2 (note that the Fano manifolds Y_n have pseudo-index 1).
Let $\pi : X = B_Z(Y) \to Y$ be a blow-up with smooth center Z between Fano manifolds X and Y. We will denote by $E = \pi^{-1}(Z)$ the exceptional divisor of π. The basic idea is very simple: we take a rational curve C in Y such that $-K_Y \cdot C = i_Y$ and we want to show that there is a rational curve $\tilde{C}$ in X, mapping surjectively to C by π, such that $-K_X \cdot \tilde{C} \leq i_Y$.
Suppose first that there is a rational curve C in Y such that $-K_Y \cdot C = i_Y$ and such that C is not contained in Z. The strict transform $\tilde{C}$ of C is a rational curve satisfying $E \cdot \tilde{C} \geq 0$ and the formula $-K_X = \pi^*(-K_Y) - (n - 1 - \dim(Z))E$ immediately implies that $-K_X \cdot \tilde{C} \leq i_Y$.
Now take a rational curve C in Y such that $-K_Y \cdot C = i_Y$ and assume $C \subset Z$. Let us decompose $N_{Z/Y\,|C}$ (we allow here a slight abuse of notations, since C might be a singular rational curve, we should rather write $\nu^*(N_{Z/Y\,|C})$ where $\nu : \mathbb{P}^1 \to C$ is the normalisation of C):

$$N_{Z/Y\,|C} = \bigoplus_{i=1}^{r} \mathcal{O}_{\mathbb{P}^1}(a_i)$$

where $r = n - \dim(Z)$. The sub-line bundles $\mathcal{O}_{\mathbb{P}^1}(a_i)$ of $N_{Y/Z\,|C}$ define rational curves $\tilde{C}_i$ of E satisfying $-K_X \cdot \tilde{C}_i = -K_Y \cdot C - (r-1)a_i$. Therefore, we are done if there exists i such that $a_i \geq 0$. Suppose the contrary, namely that $a_i \leq -1$ for all i. Then

$$-K_Z \cdot C = -K_Y \cdot C - \deg(N_{Z/Y\,|C}) \geq i_Y + \mathrm{rk}(N_{Z/Y\,|C}) = i_Y + n - \dim(Z).$$

Under the assumption $2\dim(Z) < n + i_Y - 1$, we get $-K_Z \cdot C > \dim(Z) + 1$ hence, by Mori's bend-and-break lemma (see for example [Deb01], p. 58), the curve C is numerically equivalent in Z, hence in Y, to a connected nonintegral effective rational 1-cycle (passing through 2 arbitrary fixed points of C). Each reduced irreducible component of this 1-cycle has $-K_Y$ anti-canonical degree strictly less than i_Y, contradiction! ■

Remarks: in the above proof, we only used that $-K_Y \cdot C > 0$ for any rational curve of Y. Moreover, the previous proof also shows the following. Let Y be a complex manifold, let Z be a connected submanifold of Y and let $X := B_Z(Y)$ be the blow-up of Y with center Z. Suppose both Y and X are Fano and $i_X > i_Y$. Then any rational curve C satisfying $-K_Y \cdot C = i_Y$ is contained in Z and for any such curve C, the vector bundle $N^*_{Z/Y\,|C}$ is ample.

2.2. **Proof of Theorem 2.** This proof assumes that the reader has some familiarity with Mori theory, see for example [Deb01] for a nice introduction.
By the remark above, if $i_X > i_Y$, any rational curve C in Y such that $-K_Y \cdot C = i_Y$ is contained in Z. For such a curve C, which has minimal degree with respect to an ample line bundle, its deformations in Z containing a given point cover a subvariety of dimension $\geq -K_Z \cdot C - 1$ (recall that this an easy consequence of Riemann-Roch formula and the bend-and-break lemma, see for example [Deb01], §6.5). Since the computations in the proof of Theorem 1(i) show that $-K_Z \cdot C = \dim(Z) + 1$, the deformations of C in Z containing a given point cover Z. Therefore, the Picard number of Z is one (see [Ko96], IV 3.13.3). One deduces that any rational curve C' of Z is numerically proportional (in $\mathrm{N}_1(Z)$) to C, and since C has minimal anti-canonical degree in Y, C' satisfies $-K_Z \cdot C' \geq \dim(Z) + 1$, hence $Z \simeq \mathbb{P}^{\dim(Z)}$ by [CMS00]. Finally, the computations above also show that for any line d in Z, $N_{Z/Y\,|d} \simeq \mathcal{O}_d(-1)^{\oplus n-\dim(Z)}$, which implies that $N_{Z/Y} \simeq \mathcal{O}_{\mathbb{P}^{\dim(Z)}}(-1)^{\oplus n-\dim(Z)}$ by Theorem (3.2.1) in [OSS81].
Let $E = \mathbb{P}^{\dim(Z)} \times \mathbb{P}^{n-\dim(Z)-1}$ be the exceptional divisor of π, and let ω be a Mori extremal rational curve in X such that $E \cdot \omega > 0$ (such a curve exists by the classical following argument: take any curve with strictly positive intersection with E and decompose it in the Mori cone $\mathrm{NE}(X)$ as an effective combination of extremal curves, at least one of these curves has strictly positive intersection with E). The corresponding Mori contraction φ_ω satisfies Wiśniewski's inequality [Wi91]:

$$\dim(\mathrm{Exc}(\varphi_\omega)) + \dim(f) \geq n - 1 + i_X$$

where $\mathrm{Exc}(\varphi_\omega)$ is the locus of contracted curves, and f is any non-trivial fiber of φ_ω. Since every contracted curve is proportional to ω in $\mathrm{N}_1(X)$ and since $\mathcal{O}(E)_{|E} \simeq \mathcal{O}_{\mathbb{P}^{\dim(Z)} \times \mathbb{P}^{n-\dim(Z)-1}}(-1,-1)$, none of these curves are contained in E, therefore any non-trivial fiber f of φ_ω satisfies $\dim(f) = 1$. Moreover, $i_X \geq 2$, therefore $\mathrm{Exc}(\varphi_\omega) = X$ by Wiśniewski's inequality above and φ_ω is a fibration which, by Ando's classification [An85], is a smooth $\mathbb{P}^1$-bundle over an $(n-1)$-dimensional Fano manifold X'. Moreover, $i_X = 2$, hence $i_Y = 1$, therefore $\dim(Z) = n/2$.
Let us show now that $E \cdot f = 1$ for any fiber of φ_ω. Indeed, if $\mathbb{P}^1 \to X'$ is a rational curve of X', the surface $S = \mathbb{P}^1 \times_{X'} X$ is a ruled surface, i.e., a Hirzebruch surface, and the exceptional curve of S is nothing else than $\mathbb{P}^1 \times_{X'} E$, which is a section of $S \to \mathbb{P}^1$. Hence $E \cdot f = 1$ for any fiber of φ_ω.
One immediately deduces that $\varphi_\omega : E \to X'$ is an isomorphism, hence $X = \mathbb{P}(\mathcal{E})$ for some rank 2 bundle $\mathcal{E}$ over $X' \simeq E \simeq \mathbb{P}^{n/2} \times \mathbb{P}^{n/2-1}$, and E defines a sub-line bundle

of $\mathcal{E}$. Therefore $\mathcal{E}$ splits and since $N_{E/X} \simeq \mathcal{O}_{\mathbb{P}^{n/2}\times\mathbb{P}^{n/2-1}}(-1,-1)$, one deduces that

$$X \simeq \mathbb{P}(\mathcal{O}_{\mathbb{P}^{n/2}\times\mathbb{P}^{n/2-1}} \oplus \mathcal{O}_{\mathbb{P}^{n/2}\times\mathbb{P}^{n/2-1}}(1,1)),$$

which ends the proof. ∎

3. Some comments and some more examples

3.1. **On the normal bundle of the center.** The following proposition sheds some light on the example explained in Proposition 1.

Proposition 2. *Let $\pi : X \to Y$ be a blow-up with smooth connected center Z between Fano manifolds X and Y of dimension n. Suppose moreover that the conormal bundle $N^*_{Z/Y}$ is ample. Then,*

- (i) *either $i_X \leq i_Y$,*
- (ii) *or $i_X = 2$, $i_Y = 1$, Z is a Fano manifold, X is a $\mathbb{P}^1$-bundle over the $(n-1)$-dimensional Fano manifold $\mathbb{P}(N_{Z/Y})$ and Y is a $\mathbb{P}^{n-\dim(Z)}$-bundle over Z.*

(Sketch of) proof. Suppose $i_X > i_Y$ and let denote by E the exceptional divisor of π. Since $N^*_{Z/Y}$ is ample, $-E_{|E}$ is also ample and by Grauert's criterion, E is contractible to a point. Moreover, if ω is a Mori extremal rational curve in X such that $E\cdot\omega > 0$, using the same arguments as in the proof of Theorem 2, the corresponding Mori contraction φ_ω is a $\mathbb{P}^1$-bundle over the Fano manifold $E \simeq \mathbb{P}(N_{Z/Y})$. Finally, Z is Fano by [SW90] and one has $\rho_Y + 1 = \rho_X = \rho_E + 1 = \rho_Z + 2$, hence $\rho_Y = \rho_Z + 1$. This implies that there is at least one Mori extremal curve of Y which is not contained in Z. Since π is surjective, the Mori cone $\mathrm{NE}(Y)$ is generated by the images of Mori extremal curve of X, which are contained in E, except for the fibers f of the $\mathbb{P}^1$-bundle structure $X \to E$. This implies that $\pi(f)$ is extremal in Y and the corresponding extremal contraction $\psi : Y \to W$ is a fibration. But then, the fibers of ψ have dimension less or equal to $n - \dim(Z)$, hence equal to $n - \dim(Z)$ since $-K_Y \cdot \pi(f) = n - \dim(Z) + 1$. Thereore the generic fiber of ψ is $\mathbb{P}^{n-\dim(Z)}$, and finally every fiber of ψ is $\mathbb{P}^{n-\dim(Z)}$ and meets Z transversally at exactly one point (all this is verified since on X, the extremal contraction associated to f is a $\mathbb{P}^1$-bundle). Finally, $W \simeq Z$. ∎

3.2. **Some examples.** The previous Proposition implies that if $\pi : X \to Y$ is a blow-up with smooth connected center Z between Fano manifolds X and Y with $i_X > i_Y \geq 2$, then the conormal bundle $N^*_{Z/Y}$ is not ample, althought its restriction to any rational curve of minimal $-K_Y$-degree (recall that such a curve has to be contained in Z) is ample as we saw in the proof of Theorem 1 ! It is therefore the good place to give a list of examples (communicated to me by Cinzia Casagrande) of blow-ups $\pi : X \to Y$ with smooth connected center Z between Fano manifolds X and Y with $i_X > i_Y \geq 2$.

Examples. Let a, d, r and s be positive integers, let $\mathcal{E}$ be the following rank $r+s$ vector bundle over $\mathbb{P}^a$: $\mathcal{E} = \mathcal{O}_{\mathbb{P}^a}^{\oplus r} \oplus \mathcal{O}_{\mathbb{P}^a}(d)^{\oplus s}$ and let $\mathcal{F} = \mathcal{O}_{\mathbb{P}^a}(d)^{\oplus s}$ be the rank s vector subbundle of $\mathcal{E}$ defined by the $\mathcal{O}_{\mathbb{P}^a}(d)$'s factors. Define $Y := \mathbb{P}(\mathcal{E})$ and $Z := \mathbb{P}(\mathcal{F}) \simeq \mathbb{P}^a \times \mathbb{P}^{s-1}$. This submanifold Z of Y has codimension r and normal bundle $N_{Z/Y}$ equal to $\mathcal{O}_{\mathbb{P}^a\times\mathbb{P}^{s-1}}(-d,1)^{\oplus r}$. Finally, let $X := B_Z(Y)$ be the blow-up of Y with center Z. Easy computations show that Y is Fano if and only if $a \geq rd$ and X is Fano if and only if $a \geq d$, and that under these assumptions, one has

$$i_Y = \min(r+s, 1+a-rd) \text{ and } i_X = \min(r-1, s+1, 1+a-d),$$

which leads to many examples satisfying $i_X > i_Y \geq 2$. The example of lowest dimension (namely 10) for such X and Y is given when $(a, d, r, s) = (5, 1, 4, 2)$. Let us also say that many of these examples lead to X and Y satisfying $r_X < i_X$ and $r_Y < i_Y$. In the case $s = 1$, the Fano manifolds Y's have been considered by Debarre (see [Deb01], §5.11) to construct Fano manifolds of high degree $(-K_Y)^{\dim(Y)}$.

3.3. **Minimal degree of free rational curves.** When studying Fano manifolds, one often uses *free* rational curves, which means rational curves $f : \mathbb{P}^1 \to X$ such that

$$f^*T_X \simeq \mathcal{O}_{\mathbb{P}^1}(a_1) \oplus \cdots \oplus \mathcal{O}_{\mathbb{P}^1}(a_{\dim X})$$

with all the a_i's greater or equal to 0 (see [Deb01], Chapter 4 for details). One may then introduce another invariant:

$$f_X := \min\{-K_X \cdot C \,|\, C \text{ is a free rational curve of } X\},$$

which is of great importance in Hwang and Mok's recent works. It can be interpreted as the minimal anti-canonical degree of rational curves whose deformations cover an open dense subset of X. It is an easy exercise to show that if $f : X \to Y$ is a surjective morphism between Fano manifolds X and Y, then $f_X \leq f_Y$. Of course, in any of the examples above where $i_X > i_Y$, the rational curves in Y of minimal anti-canonical degree are not free curves and their deformations do not cover any dense open subset of Y.

3.4. **A final remark on a related question.** In the above results, the assumption that both X and Y are Fano is essential: when $\pi : X \to Y$ is a blow-up with smooth connected center Z between complex manifolds X and Y, understanding on which conditions X Fano (resp. Y Fano) implies Y Fano (resp. X Fano) is a completely different question, whose study has been initiaded by Wiśniewski in [Wi91]. In particular, no condition on the dimension of the center is neither necessary nor sufficient (except of course when Z is a point, see [BCW02] for a complete classification) to get one of the implications above: the examples in §3.2, in the particular case where $rd > a \geq d$, give examples of smooth blow-up $\pi : X \to Y$ between complex manifolds X and Y with X being Fano and Y not.

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Laurent Bonavero. Institut Fourier, UMR 5582, Université de Grenoble 1, BP 74. 38402 Saint Martin d'Hères. FRANCE
e-mail : bonavero@ujf-grenoble.fr